Math 217 Fall 2025
Quiz 31 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term.

   Let V be a vector space and let $T : V \to V$ be a linear transformation.

   (a) A vector $v \in V$ is an *eigenvector* of T if ...

   > **Solution:** A vector $v \in V$ is an eigenvector of T if $v \neq 0$ and there exists a scalar $\lambda \in \mathbb{R}$ such that
   > $$T(v) = \lambda v.$$

   (b) A scalar λ is an *eigenvalue* of T if ...

   > **Solution:** A scalar $\lambda \in \mathbb{R}$ is an eigenvalue of T if there exists a nonzero vector $v \in V$ such that
   > $$T(v) = \lambda v.$$

2. Suppose V is a vector space and $\{v_1, v_2, v_3, v_4\}$ is a set of vectors that spans V. Show: If $w \neq 0$ is another vector in V, then we can find $j \in \{1, 2, 3, 4\}$ such that the set

$$\{w, v_k \mid k \in \{1, 2, 3, 4\} \setminus \{j\}\}$$

spans V.

> **Solution:** Consider the set of five vectors
> $$\{w, v_1, v_2, v_3, v_4\}.$$
> Since $\{v_1, v_2, v_3, v_4\}$ spans V, the dimension of V is at most 4. Therefore any list of 5 vectors in V is linearly dependent. Thus there exist scalars a, b_1, b_2, b_3, b_4, not all zero, such that
> $$aw + b_1v_1 + b_2v_2 + b_3v_3 + b_4v_4 = 0.$$
> Because $w \neq 0$, we may assume $a \neq 0$; otherwise we would get a nontrivial linear dependence among the v_i alone and could simply remove one of them. Solving for w,
> $$w = -\frac{b_1}{a}v_1 - \frac{b_2}{a}v_2 - \frac{b_3}{a}v_3 - \frac{b_4}{a}v_4.$$

*For full credit, please write out fully what you mean instead of using shorthand phrases.

At least one b_j is nonzero. Choose such a j. Solve instead for v_j:

$$v_j = -\frac{a}{b_j}w - \sum_{k\neq j} \frac{b_k}{b_j} v_k.$$

Thus v_j lies in the span of $\{w\} \cup \{v_k : k \neq j\}$. Since $\{v_1, v_2, v_3, v_4\}$ spans V, replacing v_j with w yields another spanning set:

$$\{w, v_k \mid k \neq j\} \text{ spans } V.$$

3. True or False. If you answer true, state TRUE. If you answer false, state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) Suppose V is a vector space, $v \in V$, and $\lambda \in \mathbb{R}$. For a linear transformation $T : V \to V$, if $T(v) = \lambda v$, then $T(T(v)) = \lambda^2 v$.

Solution: TRUE.
Using linearity,

$$T(T(v)) = T(\lambda v) = \lambda\, T(v) = \lambda(\lambda v) = \lambda^2 v.$$

(b) Suppose V is a vector space, $v \in V$, and $\lambda \in \mathbb{R}$. For a linear transformation $T : V \to V$, if $T(T(v)) = \lambda^2 v$, then $T(v) = \lambda v$.

Solution: FALSE.
Counterexample: Let $V = \mathbb{R}$, define $T(x) = -x$, and let $\lambda = 1$. Take $v = 1$. Then

$$T(T(1)) = T(-1) = 1 = \lambda^2 v.$$

But

$$T(1) = -1 \neq 1 = \lambda v.$$

Thus the implication fails.

(c) Suppose V is a vector space. The subspace

$$E_\lambda = \{v \in V : T(v) = \lambda v\}$$

is nonzero if and only if λ is an eigenvalue of T.

Solution: TRUE.
($\Rightarrow$) If E_λ contains a nonzero vector v, then $T(v) = \lambda v$ with $v \neq 0$. By definition, λ is an eigenvalue.
($\Leftarrow$) If λ is an eigenvalue, then by definition there exists a nonzero v with $T(v) = \lambda v$, so $v \in E_\lambda$ and the subspace is nonzero.

(d) Every finite dimensional inner product space $(V, \langle\, , \rangle)$ has an orthonormal basis.

Solution: TRUE.

Given any basis of V, apply the Gram–Schmidt process using the inner product $\langle\ ,\ \rangle$ to obtain an orthonormal basis of V.